Midterm Exam — Partial Differential Equations

Room 5161.0151, Tuesday 3 March 2015, 13:00 - 15:00

Duration: 2 hours

Instructions

1. The test consists of 4 questions; answer all of them.
2. The number of points for each question and each subquestion are given. 10 points are "free". The total number of points is divided by 10 to determine the final grade which will be between 1 and 10.
3. The use of books and calculators is not allowed. You may use a piece of paper with equations.

Question 1 (20 points)

Consider the equation

$$y u_x + x u_y = 0 \tag{1}$$

where $u = u(x, y)$.

a. (10 pt) Find the general solution of Eq. (1).

b. (5 pt) Find the solution of Eq. (1) with the auxiliary condition $u(0, y) = y^2$.

c. (5 pt) Sketch the characteristic curves of Eq. (1).

Solution

a. We solve the equation for the characteristic curves

$$\frac{dy}{dx} = \frac{x}{y}.$$

This equation can be separated as

$$y\,dy = x\,dx,$$

and integrated to

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C,$$

where C is the constant of integration. Solving for the integration constant we get

$$2C = y^2 - x^2.$$

Since $y^2 - x^2$ is constant along the characteristic curves we conclude that the solution of the problem has the general form

$$u(x, y) = f(y^2 - x^2),$$

where f is an arbitrary function (of one variable).

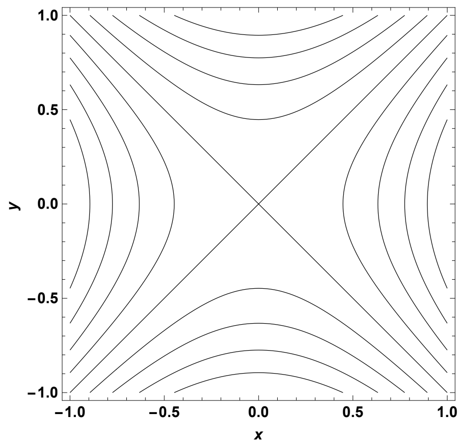
b. Applying the general solution we find

$$u(0, y) = f(y^2) = y^2.$$

Therefore $f(s) = s$, and the solution we are after is

$$u(x, y) = y^2 - x^2.$$

c. The characteristic curves are the sets $y^2 - x^2 = 2C$ for different values of C. For $C = 0$ we have $y^2 = x^2$, so $y = \pm x$. For $C \neq 0$ we get families of hyperbolas as in the picture below.



Question 2 (20 points)

Consider the second order partial differential equation

$$u_{xx} + 4u_{xy} + 3u_{yy} = 0. \tag{2}$$

a. (5 pt) Classify Eq. (2) as elliptic, hyperbolic, or parabolic.

b. (15 pt) Find a linear transformation $(x, y) \to (s, t)$ such that Eq. (2) reduces to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$, depending on its type (elliptic, hyperbolic, or parabolic).
Hint: one approach is to start by writing Eq. (2) using differential operators ∂_x, ∂_y and "completing the square".

Solution

a. We have $a_{11} = 1$, $a_{22} = 3$, and $a_{12} = 2$. Therefore

$$a_{12}^2 > a_{11}a_{22},$$

and Eq. (2) is *hyperbolic*.

b. We write Eq. (2) in the form

$$\mathcal{L}u = (\partial_x^2 + 4\partial_x\partial_y + 3\partial_y^2)u = 0.$$

Then we have

$$\mathcal{L} = \partial_x^2 + 4\partial_x\partial_y + 3\partial_y^2 = \partial_x^2 + 2\partial_x(2\partial_y) + (2\partial_y)^2 - (2\partial_y)^2 + 3\partial_y^2 = (\partial_x + 2\partial_y)^2 - \partial_y^2.$$

We want to make a transformation so that

$$\partial_s = \partial_x + 2\partial_y, \quad \partial_t = \partial_y.$$

Then

$$\mathcal{L} = \partial_s^2 - \partial_t^2,$$

and Eq. (2) becomes

$$\mathcal{L}u = u_{ss} - u_{tt} = 0,$$

which is the required form. We still need to determine the transformation.

Alternative 1
The required linear transformation has the general form

$$x = As + Bt, \quad y = Cs + Dt.$$

We have

$$u_s = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s} = Au_x + Cu_y,$$

and

$$u_t = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t} = Bu_x + Du_y.$$

Writing these relations in terms of differential operators we find

$$\partial_s = A\partial_x + C\partial_y, \quad \partial_t = B\partial_x + D\partial_y.$$

Comparing with the required relations

$$\partial_s = \partial_x + 2\partial_y, \quad \partial_t = \partial_y.$$

we find $A = 1$, $C = 2$, $B = 0$, $D = 1$. Therefore

$$x = s, \quad y = 2s + t.$$

Alternative 2
In matrix form we have

$$\begin{pmatrix} \partial_s \\ \partial_t \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}.$$

Therefore, the transformation we want is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix},$$

or

$$x = s, \quad y = 2s + t.$$

Note that in order to go from the first matrix relation to the second one we interchanged the place of the coordinates (from the left-hand side of the equation to the right-hand side and vice versa) and transposed the matrix.

Question 3 (25 points)

Consider the diffusion equation $u_t = ku_{xx}$ with $k > 0$, $0 < x < \ell$, $t > 0$, Neumann boundary conditions $u_x(0, t) = g(t)$, $u_x(\ell, t) = h(t)$, and initial data $u(x, 0) = \varphi(x)$.

a. **(5 pt)** Suppose that u_1 and u_2 are two solutions of the given diffusion equation and define $w = u_1 - u_2$. Determine the partial differential equation satisfied by w, including boundary conditions and initial data.

b. **(10 pt)** Given w from the previous subquestion, define the function

$$E(t) = \frac{1}{2}\int_0^\ell [w(x,t)]^2\, dx.$$

Show that

$$\frac{dE}{dt} \leq 0.$$

c. **(10 pt)** Use the result from subquestion **(b)** to show that for w defined in subquestion **(a)** we have $w = 0$ and therefore $u_1 = u_2$.

Solution

a. The diffusion equation is linear. Since u_1 and u_2 is a solution, so must be $w = u_1 - u_2$. Alternatively,

$$w_t = (u_1)_t - (u_2)_t = k(u_1)_{xx} - k(u_2)_{xx} = kw_{xx}.$$

Furthermore, for the boundary conditions we have

$$w_x(0,t) = (u_1)_x(0,t) - (u_2)_x(0,t) = g(t) - g(t) = 0,$$

and

$$w_x(\ell,t) = (u_1)_x(\ell,t) - (u_2)_x(\ell,t) = h(t) - h(t) = 0.$$

Finally, for the initial data we compute

$$w(x,0) = u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0.$$

Summarizing, w satisfies the diffusion equation $w_t = kw_{xx}$, with Neumann boundary conditions $w_x(0,t) = w_x(\ell,t) = 0$, and initial data $w(x,0) = 0$.

b. We compute

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\int_0^\ell w^2\, dx = \frac{1}{2}\int_0^\ell \frac{\partial}{\partial t}(w^2)\, dx = \frac{1}{2}\int_0^\ell 2ww_t\, dx = \int_0^\ell ww_t\, dx.$$

Using the fact that $w_t = kw_{xx}$ we get

$$\frac{dE}{dt} = k\int_0^\ell ww_{xx}\, dx = k\,[ww_x]_{x=0}^{x=\ell} - k\int_0^\ell (w_x)^2\, dx = -k\int_0^\ell (w_x)^2\, dx,$$

where in the last step we used the fact that $w_x|_{x=0} = w_x|_{x=\ell} = 0$. Therefore

$$\frac{dE}{dt} = -k\int_0^\ell (w_x)^2\, dx \leq 0,$$

since

$$\int_0^\ell (w_x)^2\, dx \geq 0.$$

c. We compute that

$$E(0) = \frac{1}{2}\int_0^\ell [w(x,0)]^2\,dx = \frac{1}{2}\int_0^\ell 0^2\,dx = 0.$$

Since $dE/dt \leq 0$ we conclude that E is a non-increasing function of t, therefore for $t \geq 0$ we have $E(t) \leq E(0) = 0$.
At the same time we have

$$E(t) = \frac{1}{2}\int_0^\ell [w(x,t)]^2\,dx \geq 0,$$

since it is the integral of a non-negative function.
The combination $E(t) \leq 0$ and $E(t) \geq 0$ gives $E(t) = 0$ for all $t \geq 0$. The only way for the integral

$$E(t) = \frac{1}{2}\int_0^\ell [w(x,t)]^2\,dx,$$

to be zero, given that $[w(x,t)]^2$ is a continuous function, is that

$$w(x,t) = 0$$

for all $x \in (0,\ell)$. Therefore $w = 0$ and this implies $u_1 = u_2$.

Question 4 (25 points)

Consider the eigenvalue problem $-X'' = \lambda X$ for $0 < x < \ell$ with boundary conditions $X'(0) = \alpha X(0)$ and $X'(\ell) = \alpha X(\ell)$ where $\alpha > 0$.

a. **(5 pt)** Prove that $\lambda = 0$ is not an eigenvalue.

b. **(15 pt)** Compute the positive eigenvalues for this problem.

c. **(5 pt)** Give the corresponding eigenfunction for each positive eigenvalue.

Solution

a. For $\lambda = 0$ we have the equation $X'' = 0$ with solution $X(x) = Cx + D$. Then $X'(x) = C$. The boundary condition at $x = 0$ gives

$$C = \alpha D.$$

The boundary condition at $x = \ell$ gives

$$C = \alpha(C\ell + D) \Leftrightarrow \alpha D = \alpha^2 \ell D + \alpha D \Leftrightarrow \alpha^2 \ell D = 0.$$

The only solution here is $D = 0$ which also gives $C = \alpha D = 0$ and therefore the trivial solution $X(x) = 0$ which is never an eigenfunction.

b. For $\lambda > 0$ we write $\lambda = \beta^2$, $\beta > 0$. We then have the equation $X'' + \beta^2 X = 0$ with general solution

$$X(x) = C\cos(\beta x) + D\sin(\beta x).$$

Then,

$$X'(x) = -\beta C\sin(\beta x) + \beta D\cos(\beta x).$$

At $x = 0$ we find

$$\beta D = \alpha C \Leftrightarrow C = \frac{\beta}{\alpha} D.$$

At $x = \ell$ we find

$$-\beta C\sin(\beta\ell) + \beta D\cos(\beta\ell) = \alpha C\cos(\beta\ell) + \alpha D\sin(\beta\ell)$$

and substituting the value for C we get

$$-\beta\frac{\beta}{\alpha} D\sin(\beta\ell) + \beta D\cos(\beta\ell) = \alpha\frac{\beta}{\alpha} D\cos(\beta\ell) + \alpha D\sin(\beta\ell).$$

Then we get

$$\frac{\alpha^2 + \beta^2}{\alpha} D\sin(\beta\ell) = 0.$$

The solution $D = 0$ is again rejected since then we also get $C = 0$ and the trivial solution $X(x) = 0$. We are left with

$$\sin(\beta\ell) = 0,$$

which gives the solutions

$$\beta_n = \frac{n\pi}{\ell}, \quad n = 1, 2, 3, \ldots.$$

The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, 3, \ldots.$$

c. The eigenfunction X_n for the eigenvalue λ_n is given by

$$X_n(x) = C\cos(\beta_n x) + D\sin(\beta_n x) = \frac{\beta_n}{\alpha} D\cos(\beta_n x) + D\sin(\beta_n x).$$

Substituting the value of β_n we get

$$X_n(x) = D\left[\frac{n\pi}{\ell\alpha}\cos\left(\frac{n\pi x}{\ell}\right) + \sin\left(\frac{n\pi x}{\ell}\right)\right].$$

From an eigenfunction we can always drop any constant multiplicative factor (such as D here) so, finally, the eigenfunctions are

$$X_n(x) = \frac{n\pi}{\ell\alpha}\cos\left(\frac{n\pi x}{\ell}\right) + \sin\left(\frac{n\pi x}{\ell}\right), \quad n = 1, 2, 3 \ldots.$$